

COHERENT STATES FOR THE RELATIVISTIC HARMONIC OSCILLATOR

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Abstract

Recently we have obtained, on the basis of a group approach to quantization, a Bargmann-Fock-like realization of the Relativistic Harmonic Oscillator as well as a generalized Bargmann transform relating Fock wave functions $\langle z|n \rangle$ and a set of relativistic Hermite polynomials $H_N^N(x)$, ($N = mc^2/\hbar\omega$). Nevertheless, the relativistic creation and annihilation operators satisfy typical relativistic commutation relations $[\hat{z}, \hat{z}^\dagger] \approx \text{Energy}$ (an $SL(2, R)$ algebra). Here we find higher-order polarization operators on the $SL(2, R)$ group, providing canonical creation and annihilation operators satisfying $[\hat{a}, \hat{a}^\dagger] = \hat{1}$, the eigenstates of which are "true" coherent states.

1 Group Quantization and the Relativistic Harmonic Oscillator (RHO) in the Bargmann-Fock-like realization.

The quantization of relativistic systems in a manifestly covariant way requires the use of commutation relation of the form $[\hat{x}, \hat{p}] \approx \text{Energy}$, which means a deviation from the canonical rules. If the Hamiltonian, \hat{x} and \hat{p} close a Lie algebra, it is possible to resort to some kind of group quantization method, i.e. some technique of obtaining unitary irreducible representations of a group the Lie algebra of which coincides with the Poisson algebra of the physical system. In the present case there is a Lie algebra, a central extension of $SL(2, R)$ ($SO(3, 2)$ in 3+1 dimensions):

$$[\hat{E}, \hat{x}] = -i\frac{\hbar}{m}\hat{p}, \quad [\hat{E}, \hat{p}] = im\omega^2\hbar\hat{x}, \quad [\hat{x}, \hat{p}] = i\hbar\left(\hat{1} + \frac{1}{mc^2}\hat{E}\right), \quad (1)$$

which reproduces the Poincaré algebra under the $\omega \rightarrow 0$ limit and the Newton (non-relativistic harmonic oscillator) algebra when $c \rightarrow \infty$ and that, therefore, earns to be considered as the algebra of a relativistic harmonic oscillator.

Then, our starting point will be a central pseudo-extension of the group $SL(2, R)$, denoted by $SL(2, R) \tilde{\otimes} U(1)$ [1], whose coboundary is generated by a function which is an integer power of the parameter of the Cartan subgroup. The precise techniques of the group-quantization procedure [2] will be explained on the way.

The $\tilde{G} \equiv SL(2, R) \tilde{\otimes} U(1)$ group law is:

$$\begin{aligned} z'' &= z'\eta^{-2} + z\kappa' + \frac{z}{N(1+\kappa)}(z^*z'\eta^{-2} + z'^*z\eta^2) \\ z^{*''} &= z'^*\eta^2 + z^*\kappa' + \frac{z^*}{N(1+\kappa)}(zz'^*\eta^2 + z'z^*\eta^{-2}) \\ \eta'' &= \sqrt{\frac{2}{1+\kappa''}} \left[\sqrt{\frac{1+\kappa'}{2}} \sqrt{\frac{1+\kappa}{2}} \eta' \eta + \sqrt{\frac{2}{1+\kappa'}} \sqrt{\frac{2}{1+\kappa}} \left(\frac{z'z^*}{2N} \eta' \eta^* + \frac{z'^*z}{2N} \eta'^* \eta \right) \right] \\ \zeta'' &= \zeta' \zeta (\eta'' \eta'^{-1} \eta^{-1})^{-2N}, \end{aligned} \quad (2)$$

where

$$\begin{aligned} \kappa &\equiv \sqrt{1 + \frac{2zz^*}{N}} \\ \kappa'' &= \kappa' \kappa + \frac{1}{N} (z^*z'\eta^{-2} + z'^*z\eta^2), \end{aligned}$$

and $z \in C$, $\eta \in U(1) \subset SL(2, R)$, $\zeta \in U(1)$ and $N \equiv \frac{mc^2}{\hbar\omega}$. It must be noted that N is quantized ($N = 1, 3/2, 2, 5/2, \dots$) on $SL(2, R)$ but a positive number on the Universal covering group.

The coboundary

$$\Delta \equiv (\eta'' \eta'^{-1} \eta^{-1})^{-2N} : SL(2, R) \times SL(2, R) \rightarrow U(1), \quad (3)$$

which is generated by

$$\eta^{-2N} : SL(2, R) \rightarrow U(1), \quad (4)$$

realizes a pseudo-extension. We say that Δ is a pseudo-cocycle and realizes a pseudo-extension rather than a trivial cocycle (coboundary) realizing a trivial extension because in the $c \rightarrow \infty$ limit, $(\eta'' \eta'^{-1} \eta^{-1})^{-2N}$ goes to a true cocycle on the non-relativistic harmonic oscillator (Newton) group (see [3] for a general study of the contraction process under which a true cocycle is generated by a coboundary).

Group quantization uses the (exponential of the) right-invariant vector fields, which act on $U(1)$ -equivariant complex functions on \tilde{G} as ordinary derivatives, to define a group representation (Bohr-Sommerfeld quantization). This representation is reducible, as can be stated by the existence of non-trivial operators (all the left-invariant vector fields) commuting with the representation.

The full quantization is achieved by reducing this representation in a way compatible with the action of right vector fields. The reduced Hilbert space is made of complex functions Ψ on \tilde{G} such that

$$\begin{aligned} \Psi(\zeta * g) &= \zeta \cdot \Psi(g), \quad \zeta \in U(1), g \in G \\ \hat{X}^L \Psi &= 0, \quad \forall \hat{X}^L \in \mathcal{P} \end{aligned}$$

where a Polarization \mathcal{P} is a maximal left subalgebra containing the generators in the kernel of Δ and excluding the central generator $\Xi \equiv \tilde{X}_\zeta^L$ of $U(1)$.

The left- and right-invariant vector fields are:

$$\begin{aligned}\tilde{X}_z^L &= \kappa \frac{\partial}{\partial z} + \frac{iz^*}{2N(1+\kappa)} \left(i\eta \frac{\partial}{\partial \eta} \right) - \frac{iz^*}{1+\kappa} \Xi \\ \tilde{X}_{z^*}^L &= \kappa \frac{\partial}{\partial z^*} - \frac{iz}{2N(1+\kappa)} \left(i\eta \frac{\partial}{\partial \eta} \right) + \frac{iz}{1+\kappa} \Xi \\ \tilde{X}_\eta^L &= i\eta \frac{\partial}{\partial \eta} - 2iz \frac{\partial}{\partial z} + 2iz^* \frac{\partial}{\partial z^*} \\ \tilde{X}_\zeta^L &= i\zeta \frac{\partial}{\partial \zeta} \equiv \Xi,\end{aligned}\tag{5}$$

$$\begin{aligned}\tilde{X}_z^R &= \frac{\eta^{-2}}{(1+\kappa)} \left[\frac{(1+\kappa)^2}{2} \frac{\partial}{\partial z} + \frac{z^{*2}}{N} \frac{\partial}{\partial z^*} - i \frac{z^*}{N} \left(i\eta \frac{\partial}{\partial \eta} \right) + iz^* \Xi \right] \\ \tilde{X}_{z^*}^R &= \frac{\eta^2}{(1+\kappa)} \left[\frac{(1+\kappa)^2}{2} \frac{\partial}{\partial z^*} + \frac{z^2}{N} \frac{\partial}{\partial z} + i \frac{z}{N} \left(i\eta \frac{\partial}{\partial \eta} \right) - iz \Xi \right] \\ \tilde{X}_\eta^R &= i\eta \frac{\partial}{\partial \eta} \\ \tilde{X}_\zeta^R &= i\zeta \frac{\partial}{\partial \zeta} \equiv \Xi.\end{aligned}\tag{6}$$

The operators are

$$\hat{z} \equiv \tilde{X}_{z^*}^R, \quad \hat{z}^\dagger \equiv -\tilde{X}_z^R, \quad \hat{H} \equiv \frac{i}{2} \tilde{X}_\eta^R = \frac{i}{\omega} \tilde{X}_t^R = \frac{1}{\hbar\omega} \hat{E},\tag{7}$$

where $\eta = e^{i\theta}$ and $\theta = \frac{\omega t}{2}$, with the commutation relations

$$[\hat{H}, \hat{z}] = -\hat{z}, \quad [\hat{H}, \hat{z}^\dagger] = \hat{z}^\dagger, \quad [\hat{z}, \hat{z}^\dagger] = \hat{1} + \frac{1}{N} \hat{H}\tag{8}$$

A polarization is given by $\mathcal{P} = \langle \tilde{X}_\eta^L, \tilde{X}_z^L \rangle$, with solutions

$$\begin{aligned}\Psi &= \zeta \sum_n e^{-2in\theta} c_n \Phi_n^N(z, z^*) \\ \Phi_n^N(z, z^*) &\equiv \frac{1}{\pi \sqrt{n!}} \sqrt{\frac{(2N+n-1)!}{(2N-1)!(2N)^n}} \sqrt{\frac{2N-1}{2N}} \left(\frac{1+\kappa}{2} \right)^{-N-n} z^{*n}\end{aligned}\tag{9}$$

which constitute the Fock-Bargmann-like space with the group invariant measure $\frac{dz dz^*}{\kappa} 1$.

The relativistic Fock space is given by:

$$\langle 0|0 \rangle = 1, \quad |n \rangle = \frac{(\hat{z}^\dagger)^n |0 \rangle}{\sqrt{n! \prod_{s=1}^{n-1} (1 + \frac{s}{2N})}},\tag{10}$$

¹In reality the measure on the whole group is $\frac{dz dz^* d\theta}{\kappa}$ but the time variable (or θ) can be factorized out.

$$\begin{aligned}
\hat{z}|n\rangle &= \sqrt{n(1 + \frac{n-1}{2N})}|n-1\rangle \\
\hat{z}^\dagger|n\rangle &= \sqrt{(n+1)(1 + \frac{n}{2N})}|n+1\rangle \\
\hat{H}|n\rangle &= n|n\rangle
\end{aligned} \tag{11}$$

2 Relativistic coherent states (RCS).

In the group-quantization scheme, the coherent states (generalizing the standard non-relativistic coherent states [4]), as well as the wave functions given above, are defined by mean of infinitesimal relations (differential polarization equations), rather than a finite group action on the vacuum associated with a previously given representation of the group [5,6] (see [7,8,9] for a more general study of overcomplete families of states non-necessarily associated with groups). They are defined simply as:

$$|z\rangle \equiv \sum_n \Phi_n^{N*}(z, z^*)|n\rangle \leftrightarrow \Phi_n^N(z, z^*) = \langle z|n\rangle \tag{12}$$

The associated (time-independent) wave functions $\langle z'|z\rangle \equiv \Phi_z(z')$ correspond to the choice $c_n \equiv c_n(z) = \Phi_n^N(z, z^*)$ in $\Psi(z')$.

The RCS are identified with the generalized coherent states on the unit complex disk [5] once the change of variables $z_D = \sqrt{\frac{2}{N}} \frac{z}{1+\kappa} \in D$ ($z \in C$), has been made, where D is the unit complex disk.

The expectation values of \hat{z} and \hat{z}^\dagger in the coherent states are $\langle \hat{z} \rangle \equiv \frac{\langle z|\hat{z}|z\rangle}{\langle z|z\rangle} = z$ and $\langle \hat{z}^\dagger \rangle = z^*$, making the variables $z, z^* \in C$ specially suitable to describe the Bargmann-Fock-like representation. Defining the operators \hat{x} and \hat{p} in the usual way, i.e.

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{z} + \hat{z}^\dagger), \quad \hat{p} = \sqrt{\frac{m\omega\hbar}{2}} (\hat{z}^\dagger - \hat{z}) \tag{13}$$

we get $\langle \hat{x} \rangle = x$, $\langle \hat{p} \rangle = p$, where x and p are defined in the same way, constituting the phase-space coordinates for Anti-deSitter space-time.

Repeating the group quantization in the new variables we obtain the x -representation in terms of the relativistic Hermite Polynomials [10]. Both representations are related through the Relativistic Bargmann transform [11], the kernel of which is nothing other than the configuration-space wave function of the coherent states $|z\rangle$ defined above:

$$\langle x|z\rangle = \hat{C}^N \left(\frac{1+\kappa}{2} \right)^{-N} \alpha^N \left[1 + \frac{s_0^2}{N} \right]^{-N}, \tag{14}$$

where

$$\begin{aligned}
s_0 &\equiv \sqrt{\frac{m\omega}{\hbar}} x - \frac{\sqrt{2}z\alpha}{1+\kappa}, \quad \alpha \equiv \sqrt{1 + \frac{\omega^2 x^2}{c^2}} \\
\hat{C}^N &\equiv \frac{1}{\sqrt{\pi}} \left(\frac{m\omega}{\hbar\pi} \right)^{\frac{1}{4}} \sqrt{\frac{2N-1}{2N}} \sqrt{\frac{\Gamma(N)}{\sqrt{N}\Gamma(N-\frac{1}{2})}}
\end{aligned} \tag{15}$$

In the non-relativistic limit we regain the usual coherent states in configuration space:

$$\langle x|z \rangle^{N.R.} = \frac{1}{\sqrt{\pi}} \left(\frac{m\omega}{\hbar\pi} \right)^{\frac{1}{4}} e^{-z^2/2 + \sqrt{2m\omega/\hbar} x z} e^{-\frac{1}{2} \left(\frac{m\omega}{\hbar} x^2 + |z|^2 \right)} \quad (16)$$

The uncertainty relations for the operators \hat{x} and \hat{p} are:

$$\Delta\hat{x}\Delta\hat{p} = \frac{\hbar}{2} \sqrt{\kappa^2 + \frac{1}{4N^2} [4|z|^4 - (z^2 + z^{*2})]} \geq \frac{1}{2} \hbar \kappa = \frac{1}{2} |\langle [\hat{x}, \hat{p}] \rangle| \quad (17)$$

The equality holds for $z = |z|e^{in\pi}$, i.e. $z \in R$, defining the so-called “intelligent states”, but only for $z = 0$ (the vacuum) we reach the absolute minimum (see [12] for the calculations in the unit Disk).

3 Canonical (higher-order) creation and annihilation operators: canonical, relativistic coherent states.

The definition of polarization in group quantization can be generalized so as to admit operators in the left enveloping algebra. This generalization has been already exploited in finding a position operator for the free relativistic particle [13] (as well as in solving anomalous problems [2]). In the present case it also makes sense to look for basic operators satisfying canonical (versus manifestly covariant) commutation relations. Let us then seek a power series in \tilde{X}_z^L and $\tilde{X}_{z^*}^L$,

$$\begin{aligned} \tilde{X}_{z^*}^{LHO} &= \tilde{X}_{z^*}^L + \frac{\alpha}{N} \tilde{X}_z^L \tilde{X}_z^L \tilde{X}_{z^*}^L + \dots \\ \tilde{X}_\eta^{LHO} &= \tilde{X}_\eta^L - \mu \tilde{X}_z^L \tilde{X}_{z^*}^L - \frac{\nu}{N} \tilde{X}_z^L \tilde{X}_z^L \tilde{X}_{z^*}^L \tilde{X}_{z^*}^L + \dots, \end{aligned} \quad (18)$$

such that $\mathcal{P}^{HO} = \langle \tilde{X}_\eta^{LHO}, \tilde{X}_{z^*}^{LHO} \rangle$ contains \tilde{X}_η^L and excludes \tilde{X}_ζ^L . The coefficients of the power series are determined by the requirement that \mathcal{P}^{HO} is a polarization and the corresponding right operators define a unitary action on the wave functions Ψ *which fortunately are the same as before*. More concretely,

$$\begin{aligned} [\tilde{X}_\eta^{LHO}, \tilde{X}_{z^*}^{LHO}] &= -2\tilde{X}_{z^*}^{LHO} \\ [\tilde{X}_z^{RHO}, \tilde{X}_{z^*}^{RHO}] &= \hat{1} \end{aligned} \quad (19)$$

The resulting higher-order (canonical) creation and annihilation operator are:

$$\begin{aligned} \hat{z}^{HO} &\equiv \hat{a} = \hat{z} - \left(\frac{1}{4N} - \frac{3}{32N^2} \right) \hat{z}^\dagger \hat{z} \hat{z} + \frac{7}{32N^2} \hat{z}^\dagger \hat{z}^\dagger \hat{z} \hat{z} \hat{z} + \dots \equiv \sqrt{\frac{2}{1+\hat{\kappa}}} \hat{z} \\ \hat{z}^{\dagger HO} &\equiv \hat{a}^\dagger = \hat{z}^\dagger \sqrt{\frac{2}{1+\hat{\kappa}}} \end{aligned} \quad (20)$$

and the energy operator is:

$$\hat{H}^{HO} = N(\hat{\kappa} - 1) = \hat{a}^\dagger \hat{a} \quad (21)$$

where $\hat{\kappa} \equiv \sqrt{1 + \frac{2}{N} (\hat{z}^\dagger \hat{z})}$ and the operator $\sqrt{\frac{2}{1+\hat{\kappa}}}$ must be considered as functions of the single operator $(\hat{z}^\dagger \hat{z})$.

The commutation relations,

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= \hat{1} \\ [\hat{H}^{HO}, \hat{a}] &= -\hat{a} \\ [\hat{H}^{HO}, \hat{a}^\dagger] &= \hat{a}^\dagger, \end{aligned} \quad (22)$$

have the non-relativistic (canonical) form. Their action on the Fock space is:

$$\begin{aligned} \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \\ \hat{a}^\dagger|n\rangle &= \sqrt{(n+1)}|n+1\rangle \\ \hat{H}^{HO}|n\rangle &= n|n\rangle, \end{aligned} \quad (23)$$

which reproduces the non-relativistic harmonic oscillator representation, although it must be stressed that the states $|n\rangle$ are the same relativistic energy eigenstates as before.

3.1 Canonical coherent states.

It seems quite natural to define canonical coherent states $|a\rangle$ as the eigenstates of the canonical annihilation operator, $\hat{a}|a\rangle = a|a\rangle$, with solutions:

$$|a\rangle = e^{-|a|^2/2} \sum_n \frac{a^n}{\sqrt{n!}} |n\rangle, \quad (24)$$

and define a non-relativistic Bargmann-Fock space in the usual way:

$$\langle a|n\rangle = \langle n|a\rangle^* = e^{-|a|^2/2} \frac{a^{*n}}{\sqrt{n!}} \equiv \Psi_n^{N.R.}(a) \quad (25)$$

The connection to the relativistic Bargmann-Fock space is given by

$$\begin{aligned} \Psi_a(z) \equiv \langle z|a\rangle &= \sum_n \langle z|n\rangle \langle n|a\rangle = \sum_n \Psi_n(z) \Psi_n^{N.R.}(a)^* \\ &= \frac{1}{\pi} \sqrt{\frac{2N-1}{2N}} e^{-|a|^2/2} \left(\frac{1+\kappa}{2}\right)^{-N} \sum_n \frac{1}{n!} \sqrt{\frac{(2N)_n}{(2N)^n}} \left(\frac{2az^*}{1+\kappa}\right)^n \\ &\approx \frac{1}{\pi} e^{-|a|^2/2} e^{-|z|^2/2} e^{az^*} \left\{ 1 - \frac{1}{4N} \left[1 - \frac{1}{2} (|z|^2 - az^*) (3|z|^2 - az^*) \right] + \dots \right\} \end{aligned} \quad (26)$$

The expectation value $\langle a|\hat{z}|a\rangle$ defines a classical function $z = z(a)$ relating the variables a, a^* and z, z^* as follows:

$$\langle a|\hat{z}|a\rangle = a \sum_n c_n \langle a|(\hat{a}^\dagger \hat{a})^n|a\rangle \quad (27)$$

where c_n are the coefficients of the power series of $f(u) = \sqrt{1 + \frac{u}{2N}}$. Then we define:

$$z(a) = \sqrt{1 + \frac{|a|^2}{2N}} a \quad (28)$$

Note that although $\langle a | (\hat{a}^\dagger \hat{a})^n | a \rangle \neq \langle a | \hat{a}^{\dagger n} \hat{a}^n | a \rangle = |a|^{2n}$, any operator of the form $\hat{F} = \hat{O} \hat{a}^m$ (or $\hat{G} = \hat{a}^{\dagger p} \hat{O}$), where $[\hat{H}^{HO}, \hat{O}] = 0$, defines a classical function $F(a)$ (or $G(a)$) by the formula:

$$F(a) = a^m \sum_n o_n |a|^{2n}, \quad G(a) = a^{*p} \sum_n o_n |a|^{2n}, \quad (29)$$

where $\langle a | \hat{O} | a \rangle = \sum_n o_n \langle a | (\hat{H}^{HO})^n | a \rangle$.

The functions

$$a(z) = \sqrt{\frac{2}{1 + \kappa}} z, \quad a^*(z) = \sqrt{\frac{2}{1 + \kappa}} z^* \quad (30)$$

turn out to be the Darboux coordinates taking the symplectic form $\Omega \equiv \frac{1}{\kappa} dz \wedge dz^*$ to canonical form $\Omega = da \wedge da^*$.

Finally, we define

$$\begin{aligned} \hat{q} &\equiv \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger) \\ \hat{\pi} &\equiv \sqrt{\frac{m\omega\hbar}{2}} (\hat{a}^\dagger - \hat{a}) \end{aligned} \quad (31)$$

satisfying

$$[\hat{q}, \hat{\pi}] = i\hbar \hat{1}, \quad (32)$$

and their corresponding classical functions. For these operators we obviously obtain

$$\Delta \hat{q} \Delta \hat{\pi} = \frac{\hbar}{2} \quad (33)$$

on the $|a\rangle$ states.

4 Final Remarks

The construction of the canonical (higher-order) creation and annihilation operators \hat{a}^\dagger and \hat{a} in the 1+1-D relativistic harmonic oscillator is a matter of convenience rather than a necessity since a first-order polarization, the manifestly covariant one, $\mathcal{P} = \langle \tilde{X}_\eta^L, \tilde{X}_z^L \rangle$ exists. However, the situation become quite different for the relativistic harmonic oscillator with spin, at least from a geometrical point of view. The reason is that the doubly pseudo-extended $SO(3, 2)$ (anti-de Sitter) Lie algebra, containing the commutators

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij} (\hat{1} + \frac{1}{mc^2} \hat{E}) \quad (34)$$

accounting for the mass, and the commutator

$$[\hat{J}_+, \hat{J}_-] = 2(\hat{J}_3 + j\hat{1}) \quad (35)$$

accounting for the spin, does not admit a consistent way (i.e. compatible with the rest of the symmetry) of defining two sets of first-order conjugated creation-annihilation (nor co-ordinate-momentum) operators. In other words, the system does not admit a (first-order) polarization and therefore the Hilbert space of $U(1)$ -equivariant complex functions on the group can be only partially reduced [14]. The full reduction then requires the introduction of higher-order operators in the polarization, generalizing those introduced here and accounting for proper *intrinsic spin* operators.

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